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On Approximate ℓ_1 Systems in Banach Spaces

S. J. Dilworth

Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208, U.S.A.
E-mail: dilworth@math.sc.edu

Denka Kutzarova[1]

Institute of Mathematics, Bulgarian Academy of Sciences, 1090 Sofia, Bulgaria
E-mail: denka@banmatpc.math.bas.bg

and

P. Wojtaszczyk[2]

Instityut Matematyki, Uniwersytet Warszawski, 02-907 Warsaw Banacha 2, Poland
E-mail: wojtaszczyk@mimuw.edu.pl

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Let X be a real Banach space and let $(f(n))$ be a positive nondecreasing sequence. We consider systems of unit vectors $(x_i)_{i=1}^{\infty}$ in X which satisfy $\|\sum_{i\in A} \pm x_i\| \geqslant |A| - f(|A|)$, for all finite $A \subset \mathbb{N}$ and for all choices of signs. We identify the spaces which contain such systems for bounded $(f(n))$ and for all unbounded $(f(n))$. For arbitrary unbounded $(f(n))$, we give examples of systems for which $[x_i]$ is H.I., and we exhibit systems in all isomorphs of ℓ_1 which are not equivalent to the unit vector basis of ℓ_1. We also prove that certain lacunary Haar systems in L_1 are quasi-greedy basic sequences. © 2001 Elsevier Science (USA)

[1] Current address: Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.

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Contents.

1. INTRODUCTION

The following elementary isometric characterization of the unit vector basis of ℓ_1^n provides the motivation for this paper.

THEOREM 0. *Let $(x_i)_{i=1}^n$ be unit vectors in a real Banach space X such that $\|\sum_{i=1}^n \pm x_i\| = n$ for all choices of signs. Then $\|\sum_{i=1}^n a_i x_i\| = \sum_{i=1}^n |a_i|$ for all scalars $a_1, \ldots, a_n$.*

We examine the "stability" of the above result with respect to small changes in the hypothesis. Accordingly, we study two classes of "approximate ℓ_1 systems." The following definition corresponds to the mildest possible weakening of the hypothesis.

DEFINITION 1.1. Let $(x_i)_{i \in I}$ be a sequence of unit vectors in a Banach space X (where $I = \{1, 2, \ldots, n\}$ or $I = \mathbb{N}$), and let $\mu \geqslant 0$. We say that (x_i) is a μ-approximate ℓ_1 system if

$$\left\| \sum_{i \in A} \pm x_i \right\| \geqslant |A| - \mu \tag{1}$$

for all finite sets $A \subset I$ and for all choices of signs.

A system which does not satisfy the above for any choice of μ will satisfy the following for some choice of $(f(n))$.

DEFINITION 1.2. Suppose that $(f(n))_{n=1}^\infty$ is a strictly positive nondecreasing sequence satisfying $\lim_{n \to \infty} f(n) = \infty$. Let $(x_i)_{i \in I}$ be a sequence of unit vectors in a Banach space X. We say that (x_i) is an $f(n)$-approximate ℓ_1 system if

$$\left\| \sum_{i \in A} \pm x_i \right\| \geqslant |A| - f(|A|)$$

for all finite sets $A \subset I$ and for all choices of signs.

The first two sections after the Introduction concern μ-approximate ℓ_1 systems. In Section 2 we characterize the Banach spaces which contain an infinite μ-approximate ℓ_1 system: they are precisely the spaces whose duals contain an isometric copy of L_1.

In Section 3 we examine the problem of extracting a large subsystem that is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_1^n. We show that there exists such a subsystem with finite complement. The size of the complement, however, does not depend on μ and ε alone. To show this we exhibit systems of size n for which the size of the complementary set is necessarily of order cn for any $c < 1/4$. These examples give a partial answer to a question raised by Elton.

The next two sections concern $f(n)$-approximate ℓ_1 systems. In Section 4 we characterize the Banach spaces which contain $f(n)$-approximate ℓ_1 systems for all $(f(n))$: they are precisely the spaces which have a spreading model isometric to the the unit vector basis of ℓ_1. We use this result to give examples of $f(n)$-approximate ℓ_1 systems whose closed linear spans do not contain any unconditional basic sequence.

In Section 5 we exhibit nontrivial examples of $f(n)$-approximate ℓ_1 systems in the space ℓ_1 (and all its isomorphs). In particular, we construct, for any given $(f(n))$, examples of both conditionally basic and unconditionally basic $f(n)$-approximate ℓ_1 systems in ℓ_1 which are not equivalent to the unit vector basis of ℓ_1.

The results which we present in Section 5 are based on some observations about "lacunary Haar" systems in L_1 and H_1. In Section 6 we pursue these ideas, proving that there is a lacunary Haar system in L_1 which is a quasi-greedy basis for its linear span.

We use standard Banach space notation and terminology throughout. For clarity, however, we recall here the notation that is used most heavily. Let X be a real Banach space with *dual space* X^*. The *unit ball* of X is the set $B(X) = \{x \in X : \|x\| \leqslant 1\}$. A subspace Y of X is said to be *complemented* if Y is the range of a continuous linear projection on X.

Let (x_n) be a sequence in X. The closed linear span of (x_n) is denoted $[x_n]$. We say that a sequence (x_n) of nonzero vectors is *basic* if there exists a positive constant K such that

$$\left\| \sum_{i=1}^{m} a_i x_i \right\| \leqslant K \left\| \sum_{i=1}^{n} a_i x_i \right\|$$

for all scalars (a_i) and all $1 \leqslant m \leqslant n \in \mathbb{N}$; (x_n) is *monotone* if we can take $K = 1$; (x_n) is α-*unconditional* if

$$\left\| \sum_{i=1}^{n} \varepsilon_i a_i x_i \right\| \leqslant \alpha \left\| \sum_{i=1}^{n} a_i x_i \right\|$$

for all scalars (a_i), all choices of signs $\varepsilon_i = \pm 1$, and all $n \geqslant 1$. We say that a sequence (y_k) of nonzero vectors is a *block basic sequence* with respect to (x_n) if there exist integers $0 = n_0 < n_1 < \cdots$ and scalars (a_i) such that

$$y_k = \sum_{i=n_{k-1}+1}^{n_k} a_i x_i \qquad (k \geqslant 1).$$

For $1 \leqslant p < \infty$, ℓ_p is the space of real sequences (a_i) equipped with the norm $\|(a_i)\|_p = (\sum_{i=1}^{\infty} |a_i|^p)^{1/p}$. The space of sequences converging to zero (resp. bounded) equipped with the supremum norm is denoted c_0 (resp. ℓ_∞). The *summing basis* of c_0 is the basis $e_1 = (1, 0, 0, \ldots)$, $e_2 = (1, 1, 0, 0, \ldots)$, etc.

Finally, it is worth emphasizing that we consider only *real* Banach spaces in this paper.

2. μ-APPROXIMATE ℓ_1 SYSTEMS

First we characterize the Banach spaces which contain an infinite μ-approximate ℓ_1 system for some $\mu > 0$. In this regard, note that Theorem 0 tells us that X contains an infinite 0-approximate ℓ_1 system if and only if X contains an isometric copy of ℓ_1.

For $A, B \subset \mathbb{N}$ and $n \in \mathbb{N}$, we write $A < B$ (respectively, $A < n$) if $\max\{i: i \in A\} < \min\{i: i \in B\}$ (respectively, $\max\{i: i \in A\} < n$).

THEOREM 2.1. *Suppose that $(x_i)_{i=1}^{\infty}$ is a μ-approximate ℓ_1 system. Then, given any decreasing sequence (ε_i) of positive numbers, there is a subsequence (y_i) of (x_i) such that*

$$\left\| \sum_{i=1}^{\infty} a_i y_i \right\| \geqslant \sum_{i=1}^{\infty} (1-\varepsilon_i)\, |a_i| \tag{2}$$

for all $(a_i) \in \ell_1$.

Proof. Set $n_0 = 0$ and suppose that $n_0 < n_1 < \cdots < n_k$ have been chosen to satisfy the following recursive hypothesis. For each choice of signs (η_i), and for all finite $A > n_k$, there exists $x^* \in B(X^*)$ such that

$$x^*(\eta_i x_{n_i}) \geqslant 1 - \varepsilon_i \qquad (1 \leqslant i \leqslant k) \tag{3}$$

and

$$x^*\left(\sum_{i=1}^{k} \eta_i x_{n_i} + \sum_{i \in A} \eta_i x_i \right) \geqslant k + |A| - \mu. \tag{4}$$

For $k=0$, note that (3) is vacuously true and that (4) just follows from the Hahn–Banach theorem and the definition of a μ-approximate ℓ_1 system. So the recursive definition starts. The following claim will establish the inductive step.

Claim. There exists $n_{k+1} > n_k$ such that if $A \geqslant n_{k+1}$ and (η_i) is any choice of signs then there exists $x^* \in B(X^*)$ satisfying (3), (4), and

$$x^*(\eta_i x_i) > 1 - \varepsilon_{k+1} \qquad (i \in A). \tag{5}$$

Proof of Claim. We argue for a contradiction. If not, then there exists a choice of signs (η_i) and an infinite sequence (A_j), with $n_k < A_1 < \cdots < A_j < \cdots$, such that, for each $j \geqslant 1$, whenever (3) and (4) are satisfied for $A = A_j$ then (5) is not satisfied, i.e.

$$\min_{i \in A_j} x^*(\eta_i x_i) \leqslant 1 - \varepsilon_{k+1}. \tag{6}$$

Fix $N \geqslant 1$. By the recursive hypothesis there exists $x^* \in B(X^*)$ satisfying (3) and

$$x^*\left(\sum_{i=1}^{k} \eta_i x_{n_i} + \sum_{j=1}^{N} \sum_{i \in A_j} \eta_i x_i\right) \geqslant k + \left(\sum_{j=1}^{N} |A_j|\right) - \mu. \tag{7}$$

Thus (4) is satisfied by x^* for $A = A_j$ and for each $1 \leqslant j \leqslant N$. Hence (6) is satisfied for each $1 \leqslant j \leqslant N$. This implies

$$x^*\left(\sum_{i=1}^{k} \eta_i x_{n_i} + \sum_{j=1}^{N} \sum_{i \in A_j} \eta_i x_i\right) \leqslant k + \sum_{j=1}^{N} |A_j| - N\varepsilon_{k+1}. \tag{8}$$

But (8) contradicts (7) when $N > \mu/\varepsilon_{k+1}$.

For n_{k+1} as given by the claim, the recursive hypothesis is satisfied for $k+1$. Let $y_i = x_{n_i}$. Fix $(a_i) \in \ell_1$. Let $\eta_i = \operatorname{sgn}(a_i)$. Then by (3) and a weak compactness argument there exists $x^* \in B(X^*)$ such that $x^*(\eta_i y_i) \geqslant 1 - \varepsilon_i$ for all $i \geqslant 1$. Thus

$$\left\| \sum_i a_i y_i \right\| \geqslant x^*\left(\sum_i a_i y_i\right) \geqslant \sum (1-\varepsilon_i)\,|a_i|. \quad \blacksquare$$

Recall that a normalized sequence (y_i) which satisfies (2) for some sequence (ε_i) of positive numbers decreasing to zero is called *an asymptotically isometric copy of* ℓ_1. This class of sequences was introduced by Hagler [11] and has been used recently by Dowling and Lennard [8] in fixed point theory.

THEOREM 2.2. *Let X be a Banach space. The following are equivalent:*

(a) *X contains an infinite μ-approximate ℓ_1 system for some $\mu \geqslant 0$.*

(b) *X contains an asymptotically isometric copy of ℓ_1.*

(c) *X^* contains an isometric copy of $L_1[0,1]$.*

Proof. Theorem 2.1 yields (a) $\Rightarrow$ (b). Suppose that (x_i) is an asymptotically isometric copy of ℓ_1 in X which satisfies (2) for some sequence (ε_i) of positive numbers decreasing to zero. By passing to a subsequence we may assume that $\sum_{i=1}^{\infty} \varepsilon_i < \infty$. This implies that (x_i) is a μ-approximate ℓ_1 system for $\mu = \sum_{i=1}^{\infty} \varepsilon_i$, so (b) $\Rightarrow$ (a). The equivalence of (b) and (c) was proved in [11] (cf. also [6] for the complex version). ▮

Remark 2.3. For several further equivalences see [6, 11].

We do not know, however, if a μ-approximate ℓ_1 system is *itself* an asymptotically isometric copy of ℓ_1.

QUESTION 2.4. *Suppose that (x_i) is a μ-approximate ℓ_1 system. Does there exist a sequence (ε_i) of positive numbers decreasing to zero such that*

$$\left\| \sum_{i=1}^{\infty} a_i x_i \right\| \geqslant \sum_{i=1}^{\infty} (1-\varepsilon_i)\, |a_i|.$$

for all $(a_i) \in \ell_1$?

The following "global" result will be used in Section 6.

PROPOSITION 2.5. *Let (x_i) be a basic μ-approximate ℓ_1 system with basis constant K. Then*

$$\min\left(\frac{1}{6}, \frac{1}{24K\mu}\right) \sum_{i=1}^{\infty} |a_i| \leqslant \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leqslant \sum_{i=1}^{\infty} |a_i| \tag{9}$$

for all $(a_i) \in \ell_1$.

Proof. Suppose that $\sum_{i=1}^{n} |a_i| = 1$. Let $\eta_i = \operatorname{sgn} a_i$ $(1 \leqslant i \leqslant n)$. There exists $x^* \in Ba(X^*)$ such that $x^*(\sum_{i=1}^{n} \eta_i x_i) \geqslant n - \mu$. Let $A = \{1 \leqslant i \leqslant n : \eta_i x^*(x_i) \leqslant 3/4\}$. Then $(n-|A|) + (3/4)\,|A| \geqslant n - \mu$, i.e. $|A| \leqslant 4\mu$. We now consider two cases. For the first case suppose that $\sum_{i \in A} |a_i| < 1/3$. Then

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \geqslant \sum_{i \notin A} a_i x^*(x_i) - \sum_{i \in A} |a_i| > \frac{2}{3}\frac{3}{4} - \frac{1}{3} = \frac{1}{6}.$$

For the second case suppose that $\sum_{i\in A}|a_i|\geqslant 1/3$. Then, since (x_i) is basic with basis constant K, we have

$$\left\|\sum_{i=1}^{n} a_i x_i\right\|\geqslant\frac{1}{2K}\max_{i\in A}|a_i|\geqslant\frac{1}{2K}\frac{\sum_{i\in A}|a_i|}{|A|}\geqslant\frac{1}{2K}\frac{1/3}{4\mu}=\frac{1}{24K\mu}. \quad\blacksquare$$

Remark 2.6. The factor $1/\mu$ in (9) is best possible up to a multiplicative constant. To see this, let $(x_i)_{i=1}^{\mu+1}$ be the usual basis of $\ell_\infty^{\mu+1}$, where $\mu\in\mathbb{N}$. Then $(x_i)_{i=1}^{\mu+1}$ is a μ-approximate ℓ_1 system for which $\|1/(\mu+1)\sum_{i=1}^{\mu+1}x_i\|=1/(\mu+1)$.

3. ALMOST ISOMETRIC RESULTS

Our goal in this section is to understand how well a μ-approximate ℓ_1 system compares with the standard unit vector basis of ℓ_1. The following result shows that, given $\varepsilon>0$ and a μ-approximate ℓ_1 system, one can obtain a $(1+\varepsilon)$-isometric copy of the unit vector basis of ℓ_1 by removing a *finite* set of vectors from the system.

PROPOSITION 3.1. *Suppose that (x_i) is a μ-approximate ℓ_1 system. Then, given $\varepsilon>0$, there exists a finite set A such that*

$$\left\|\sum_{i\in B}\pm x_i\right\|\geqslant|B|-\varepsilon \tag{10}$$

whenever $A\cap B=\varnothing$. In particular, if $0<\varepsilon<1$, then

$$(1-\varepsilon)\sum_{i\in B}|a_i|\leqslant\left\|\sum_{i\in B}a_i x_i\right\|\leqslant\sum_{i\in B}|a_i| \tag{11}$$

for all scalars (a_i) whenever $A\cap B=\varnothing$.

Proof. We may assume that μ is the least constant satisfying (1). There exist $N\geqslant 1$ and a choice of signs $(\eta_i)_{i=1}^{N}$ such that

$$\left\|\sum_{i=1}^{N}\eta_i x_i\right\|<N+\varepsilon-\mu.$$

Let $(\eta_i)_{N+1}^{\infty}$ be any choice of signs. Then, for $n>N$, we have

$$\begin{aligned}\left\|\sum_{i=N+1}^{n}\eta_i x_i\right\|&\geqslant\left\|\sum_{i=1}^{n}\eta_i x_i\right\|-\left\|\sum_{i=1}^{N}\eta_i x_i\right\|\\&\geqslant(n-\mu)-(N+\varepsilon-\mu)=(n-N)-\varepsilon.\end{aligned}$$

Taking $A = \{1, \ldots, N\}$ gives (10), and (11) follows from (10) by the triangle inequality. ∎

The following example shows that the cardinality of a set A which satisfies (10) does not depend only on ε and μ, even when (x_i) is 1-unconditional. (Recall that (x_i) is α-unconditional if $\|\sum_{i=1}^{\infty} \pm a_i x_i\| \leqslant \alpha \|\sum_{i=1}^{\infty} a_i x_i\|$ for all scalars (a_i) and for all choices of signs.)

EXAMPLE 3.2. For each $n \geqslant 1$, let $(e_i)_{i=1}^n$ be the unit vector basis of ℓ_p^n, where p is chosen so that $n^{1/p} = n-1$. Then $(e_i)_{i=1}^n$ is a 1-approximate ℓ_1 system. But it is clear that if A satisfies (10) for $\varepsilon = 1/2$, then $|A| \to \infty$ as $n \to \infty$.

For (11), on the other hand, the cardinality of A depends only on ε and μ, provided that (x_i) is 1-unconditional.

PROPOSITION 3.3. (a) *Let (x_i) be a μ-approximate ℓ_1 system. There exists a nonnegative sequence (ε_i) such that $\sum_{i=1}^{\infty} \varepsilon_i \leqslant \mu$ and*

$$\left\| \sum_{i=1}^{\infty} a_i x_i \right\| \geqslant \sum_{i=1}^{\infty} a_i(1-\varepsilon_i) \tag{12}$$

for all $(a_i) \in \ell_1$.

(b) *Let $\varepsilon > 0$. If $(x_i)_{i=1}^{\infty}$ is 1-unconditional then A can be chosen to satisfy (11) with $|A| \leqslant \lfloor \mu/\varepsilon \rfloor$.*

Proof. (a) For each $n \in I$, select $x_n^* \in B(X^*)$ such that $x_n^*(\sum_{i=1}^n x_i) \geqslant n-\mu$. Note that, if $m \leqslant n$, then $x_n^*(\sum_{i=1}^m x_i) \geqslant m-\mu$. By passing to a subsequence, we may assume that $x_n^*(x_m) \to 1-\varepsilon_m$ as $n \to \infty$. Thus, for each $m \in I$,

$$\sum_{i=1}^{m} (1-\varepsilon_i) = \lim_{n\to\infty} x_n^* \left(\sum_{i=1}^{m} x_i \right) \geqslant m - \mu.$$

Hence $\sum_{i=1}^{\infty} \varepsilon_i \leqslant \mu$. If $(a_i) \in \ell_1$, then

$$\left\| \sum_{i=1}^{\infty} a_i x_i \right\| \geqslant \lim_{n\to\infty} x_n^* \left(\sum_{i=1}^{n} a_i x_i \right) = \sum_{i=1}^{\infty} a_i(1-\varepsilon_i).$$

(b) For $\varepsilon > 0$, let $A = \{i \in I : \varepsilon_i > \varepsilon\}$. Then $|A| \leqslant \lfloor \mu/\varepsilon \rfloor$. If B is disjoint from A then (12) and 1-unconditionality give (11). ∎

Recall that a sequence (y_i) is *suppression* 1-*unconditional* if, whenever $A \subseteq B \subseteq I$, then $\|\sum_A a_i y_i\| \leqslant \|\sum_B a_i y_i\|$ for all scalars (a_i). We shall now show that Proposition 3.3(b) does not hold if "1-unconditional" is replaced by "suppression 1-unconditional". The following theorem is a "local" formulation of this fact.

THEOREM 3.4. *Let* $\alpha \in (0, 1/4)$ *and* $\mu \in (0, 1)$. *If* n *is a power of* 2 *then there exists a norm* $\|\cdot\|$ *on* $\mathbb{R}^n$ *with the following properties:*

(i) $(e_i)_{i=1}^n$ *is a suppression* 1*-unconditional normalized basis of* $(\mathbb{R}^n, \|\cdot\|)$.

(ii)

$$\left\| \sum_{i=1}^{n} \pm e_i \right\| \geqslant n - \mu \tag{13}$$

for all choices of signs.

(iii) *For every* $A \subseteq \{1, ..., n\}$, *with* $|A| = 1 + \lceil (3/4 + \alpha) n \rceil$, *there exists a nonzero vector* x, *with* $\operatorname{supp} x \subseteq A$, *such that*

$$\|x\|_1 \geqslant \left(1 + \frac{4\alpha\mu}{3 + 8\alpha - 4\alpha\mu}\right) \|x\|. \tag{14}$$

Let $I = \{1, 2, ..., n\}$. Since n is a power of 2, there exist n sets $S_i \subseteq I$ $(1 \leqslant i \leqslant n)$ such that (here $\triangle$ denotes the *symmetric difference*)

$$|S_i \,\Delta S_j| = |S_i \,\Delta (I \setminus S_j)| = n/2 \qquad (i \neq j).$$

Indeed, one can simply take the S_i's (under the obvious correspondence) to be the rows of the Hadamard matrix of order n (see, e.g., [5]).

For $1 \leqslant i \leqslant n$, we say that a set $S \subseteq I$ is *i-large* if either $|S \triangle S_i| < n/4$ or $|(I \setminus S) \triangle S_i| < n/4$. Note that, for each $1 \leqslant i \leqslant n$, the collection of all i-large sets is closed under complementation.

First we prove that every $S \subseteq I$ is i-large for at most one value i. So suppose that S is i_0-large and that $j \neq i_0$. Then either $|S \,\Delta S_{i_0}| < n/4$ or $|S \,\Delta (I \setminus S_{i_0})| < n/4$. We shall assume that the former holds (the proof in the latter case is similar). Since $|S_{i_0} \,\Delta S_j| = n/2$, the triangle inequality gives

$$|S \,\Delta S_j| \geqslant |S_{i_0} \,\Delta S_j| - |S \Delta\, S_{i_0}| > \frac{n}{2} - \frac{n}{4} = \frac{n}{4}.$$

Similarly, $|(I \setminus S) \,\Delta S_j| > n/4$. Thus, S is *not* j-large.

Let $y = (y_i)_{i \in I}$ be a vector whose coordinates belong to the interval $[-1, 1]$. We set $P(y) = \{i \in I : y_i > 1 - \mu\}$ and $N(y) = \{i \in I : y_i < -1 + \mu\}$. For $S \subseteq I$, we say that y is *S-admissible* and that y *is obtained from* S if the following conditions hold:

(a) $|y_i| \leqslant 1 - \mu$ whenever S is i-large.

(b) $P(y) \subseteq S$ and $N(y) \subseteq I \setminus S$.

Note that if y is S-admissible then $-y$ is $(I\setminus S)$-admissible. This follows from the fact that the collection of i-large sets is closed under complementation.

A vector y is said to be *admissible* if y is S-admissible for some $S\subseteq I$. Let F denote the collection of all admissible vectors. Then F is symmetric, i.e., if $y\in F$ then $-y\in F$. Now we can define the norm $\|\cdot\|$:

$$\left\|\sum_{i\in I} x_i e_i\right\| = \max_{y\in F} \sum_{i\in I} x_i y_i. \tag{15}$$

The symmetry of F guarantees that (15) defines a norm. The fact that this norm is suppression 1-unconditional is an immediate consequence of the following easily checked property of F: if $y\in F$ and z is obtained from y by replacing some of the coordinates of y by zeros, then $z\in F$. It is also easy to check that $\|e_i\|=1$ for all $1\leqslant i\leqslant n$.

From now on the proof is similar to that of [7]. We include it here for the sake of completeness.

Proof of (ii). Let $\eta=(\eta_i)_{i=1}^n$ be a choice of signs. Define $y=(y_i)$ thus:

$$y_i=\begin{cases}\eta_i & \text{if } P(\eta) \text{ is } \textit{not } i\text{-large},\\ (1-\mu)\,\eta_i & \text{if } P(\eta) \text{ is } i\text{-large}.\end{cases}$$

Clearly, y is $P(\eta)$-admissible, so $y\in F$. Since $P(\eta)$ is i-large for at most one index i_0, we have

$$\left\|\sum_{i=1}^n \eta_i e_i\right\| \geqslant \sum_{i=1}^n \eta_i y_i \geqslant \left(\sum_{i=1}^n \eta_i^2 - 1\right) + (1-\mu)\,\eta_{i_0}^2 = n-\mu,$$

which proves (13). ∎

Proof of (iii). Suppose $A\subset I$ with $|A|=1+\lceil(3/4+\alpha)\,n\rceil$. Choose $i_0\in A$ and let $\tilde{A}=A\setminus\{i_0\}$ (so that $|\tilde{A}|=\lceil(3/4+\alpha)\,n\rceil$). We define a vector x, with $\operatorname{supp} x=A$, thus:

$$x_i=\begin{cases}|\tilde{A}|-(3/4)\,n & \text{for } i=i_0,\\ 1 & \text{for } i\in\tilde{A}\cap S_{i_0},\\ -1 & \text{for } i\in\tilde{A}\cap(I\setminus S_{i_0}),\\ 0 & \text{otherwise.}\end{cases}$$

Now let us show that $\|x\|$ satisfies (14). Let y be an admissible vector that is obtained from $S\subseteq I$. Suppose that

$$|\tilde{A}\cap S_{i_0}\cap P(y)|+|\tilde{A}\cap(I\setminus S_{i_0})\cap N(y)|>3n/4. \tag{16}$$

Since $P(y) \subseteq S$ and $N(y) \subseteq I \setminus S$, we have

$$|S_{i_0} \cap S| + |(I \setminus S_{i_0}) \cap (I \setminus S)| > 3n/4.$$

Thus,

$$|S_{i_0} \Delta S| < n/4.$$

So S is i_0-large. Hence $|y_{i_0}| \leqslant 1 - \mu$, and so

$$\begin{aligned} \sum_{i \in I} x_i y_i &= x_{i_0} y_{i_0} + \sum_{i \in \tilde{A}} x_i y_i \\ &\leqslant (|\tilde{A}| - (3/4)\, n)(1-\mu) + |\tilde{A}|. \end{aligned} \tag{17}$$

On the other hand, if (16) does not hold, then

$$\begin{aligned} \sum_{i \in I} x_i y_i &\leqslant |x_{i_0}| + |\tilde{A}| - (|\tilde{A}| - (3/4)\, n)\, \mu \\ &\leqslant (|\tilde{A}| - (3/4)\, n)(1-\mu) + |\tilde{A}|. \end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\|x\| \leqslant |\tilde{A}| + (1-\mu)(|\tilde{A}| - (3/4)\, n). \tag{19}$$

But

$$\begin{aligned} \|x\|_1 &= |x_{i_0}| + |\tilde{A}| \\ &= (|\tilde{A}| - (3/4)\, n) + |\tilde{A}| \\ &\geqslant \|x\| + \mu(|\tilde{A}| - (3/4)\, n) \end{aligned}$$

(by (19))

$$\geqslant \left(1 + \mu\left((1-\mu) + \frac{|\tilde{A}|}{|\tilde{A}| - (3n/4)}\right)^{-1}\right) \|x\|$$

(by (19) again)

$$\geqslant \left(1 + \mu\left((1-\mu) + \frac{3+4\alpha}{4\alpha}\right)^{-1}\right) \|x\|$$

(since $|\tilde{A}| \geqslant (3/4 + \alpha)\, n$)

$$= \left(1 + \frac{4\alpha\mu}{3 + 8\alpha - 4\alpha\mu}\right) \|x\|,$$

which proves (14). ∎

Remark 3.5. The construction of the norm in Theorem 3.4 is explicit and determinstic. A "random" argument can be given (see [7] for the details) to extend Theorem 3.4 to all A satisfying $|A| > (1/2+\alpha)\, n$, although this argument has the defect that it does not give the norm explicitly. By the "almost isometric" part of a theorem of Elton [9, Theorem 1], it is *not* possible to extend the result to sets with $|A| < (1/2-\alpha)\, n$.

4. $f(n)$-APPROXIMATE ℓ_1 SYSTEMS

Henceforth $(f(n))_{n=1}^{\infty}$ will denote a strictly positive nondecreasing sequence satisfying $\lim_{n\to\infty} f(n) = \infty$. Let us first observe that the norm of the linear span of an $f(n)$-approximate ℓ_1 system does indeed behave like the ℓ_1 norm for moderately sized coefficients. It is convenient here to extend the definition of f to $\mathbb{R}^+$ by taking $f(x) = f(\lceil x \rceil)$

PROPOSITION 4.1. *Let (x_i) be an $f(n)$-approximate ℓ_1 system and suppose that $0 < \delta < M < \infty$. Then, for all scalars (a_i) satisfying $\delta \leqslant |a_i| \leqslant M$, we have*

$$\left\| \sum_{i \in A} a_i x_i \right\| \geqslant \sum_{i \in A} |a_i| - M f\left(\left(\sum_{i \in A} |a_i| \right) \Big/ \delta \right).$$

Proof. Let $\eta_i = \operatorname{sgn} a_i$. Then, by the triangle inequality,

$$\begin{aligned} \left\| \sum_{i \in A} a_i x_i \right\| &= \left\| \sum |a_i|\, \eta_i x_i \right\| \\ &\geqslant M \left\| \sum_{i \in A} \eta_i x_i \right\| - \left(\sum_{i \in A} (M - |a_i|) \right) \\ &\geqslant M(|A| - f(|A|)) - M\,|A| + \sum_{i \in A} |a_i| \\ &\geqslant \sum_{i \in A} |a_i| - M f\left(\left(\sum_{i \in A} |a_i| \right) \Big/ \delta \right). \end{aligned}$$

The last inequality follows from the fact that $\min |a_i| \geqslant \delta$, which implies $|A| \leqslant (\sum_{i \in A} |a_i|)/\delta$. ∎

We now aim to characterize the Banach spaces which contain an infinite $f(n)$-approximate ℓ_1 system for slowly increasing $f(n)$. To that end let us recall the notion of *spreading model* (see, e.g., [3]). Let (x_i) be a sequence in a Banach space X and let (s_i) be a basis for a Banach space $(Y, |\cdot|)$.

Then $(Y, |\cdot|)$ is said to be a spreading model for (x_i) if, for all $k \geqslant 1$ and for all scalars $a_1, \ldots, a_k$, we have

$$\left|\sum_{i=1}^{k} a_i s_i\right| = \lim_{\substack{n_1 \to \infty \\ n_1 < \cdots < n_k}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|.$$

Recall that every normalized sequence has a subsequence (x_i) which has a basic spreading model.

PROPOSITION 4.2. *Suppose that $(x_i)_{i=1}^{\infty}$ is a normalized sequence which has spreading model isometrically equivalent to the unit vector basis of ℓ_1. Then, given $(f(n))$, (x_i) has a subsequence (y_i) which is an $f(n)$-approximate ℓ_1 system.*

Proof. Set $n_0 = 0$. For $k \geqslant 1$, select $n_k > n_{k-1}$ with $f(n_k) > 2k$. Since (s_i) (the basis of the spreading model) is isometrically equivalent to the unit vector basis of ℓ_1, we may choose a strictly increasing sequence $(m_k)_{k=1}^{\infty}$ of positive integers such that if $1 \leqslant n \leqslant n_{k+1}$ and $m_k < j_1 < \cdots < j_n$, then

$$\left\|\sum_{i=1}^{n} \pm x_{j_i}\right\| \geqslant n - (f(n_k) - 2k).$$

Set $y_k = x_{m_k}$. Suppose that $k \geqslant 0$, that $n_k < n \leqslant n_{k+1}$, and that $1 \leqslant j_1 < \cdots < j_n$. Then

$$\begin{aligned}\left\|\sum_{i=1}^{n} \pm y_{j_i}\right\| &\geqslant \left\|\sum_{j=k+1}^{n} \pm y_j\right\| - k \\ &\geqslant (n-k) - (f(n_k) - 2k) - k \\ &= n - f(n_k) \geqslant n - f(n). \quad \blacksquare\end{aligned}$$

PROPOSITION 4.3. *Suppose that $(x_i)_{i=1}^{\infty}$ is an $f(n)$-approximate ℓ_1 system, where $\inf_{n \geqslant 1} f(n)/n = 0$. Then the spreading model associated to (x_i) is isometrically equivalent to the unit vector basis of ℓ_1.*

Proof. Fix $\varepsilon \in (0, 1)$ and $N \geqslant 1$. It suffices to show that there exists $M \in \mathbb{N}$ such that if $M \leqslant n_1 < \cdots < n_N$ then $\|\sum_{i=1}^{N} \pm x_{n_i}\| \geqslant N - \varepsilon$ for all choices of signs (for then it follows that $\|\sum_{i=1}^{N} a_i x_{n_i}\| \geqslant (1-\varepsilon) \sum_{i=1}^{N} |a_i|$ for all real scalars (a_i)). Suppose, to derive a contradiction, that there is no such M. Then there exist finite sets A_i $(i = 1, 2, \ldots)$, with $|A_i| = N$ and with $A_1 < A_2 < \cdots$, and there exists a choice of signs (η_j) such that $\|\sum_{j \in A_i} \eta_j x_j\| < N - \varepsilon$ for each i. Thus, for each $k \geqslant 1$, we have

$$\begin{aligned} Nk - f(Nk) &\leqslant \left\| \sum_{i=1}^{k} \sum_{j \in A_i} \eta_j x_j \right\| \\ &\leqslant \sum_{i=1}^{k} \left\| \sum_{j \in A_i} \eta_j x_j \right\| \\ &\leqslant \sum_{i=1}^{k} (N - \varepsilon) = Nk - k\varepsilon. \end{aligned}$$

So $f(Nk) \geqslant \varepsilon k$ for all k, which contradicts the fact that $(f(n))$ is a positive nondecreasing sequence satisfying $\inf_{n \geqslant 1} f(n)/n = 0$. ∎

Combining the previous two results, we obtain the following characterization of Banach spaces which contain $f(n)$-approximate ℓ_1 systems for all $(f(n))$.

THEOREM 4.4. *Let X be a Banach space. The following are equivalent:*

(i) *X has a spreading model isometrically equivalent to the unit vector basis of ℓ_1.*

(ii) *X contains an $f(n)$-approximate ℓ_1 system for some $(f(n))$ satisfying* $\inf_{n \geqslant 1} f(n)/n = 0$.

(iii) *For all $(f(n))$, X contains a basic $f(n)$-approximate ℓ_1 system.*

Proof. (iii) $\Rightarrow$ (ii) is clear and (ii) $\Rightarrow$ (i) is Proposition 4.3. To show (i) $\Rightarrow$ (iii), let (x_i) have spreading model isometrically equivalent to the usual ℓ_1 basis. If (x_i) has no weakly convergent subsequence, then (x_i) has a basic subsequence [18]. If (x_i) has a weakly convergent subsequence, then we may assume, after passing to a subsequence, that $y_i = x_{2i} - x_{2i-1}$ is weakly null and hence has a basic subsequence [4]. Clearly, $(y_i/2)$ also has spreading model isometrically equivalent to the ℓ_1 basis. So we can take (x_i) to be basic. Arguing as in Proposition 4.2, we see that (x_i) has a subsequence that is an $f(n)$-approximate ℓ_1 system. ∎

Remark 4.5. Beauzamy and Lapresté [3] characterized the existence of a spreading model *isomorphically* equivalent to the unit vector basis of ℓ_1.

The following renorming result is now an immediate consequence of a deep result of Odell and Schlumprecht [17, Corollary 3.3] and of the equivalence of (i) and (ii) in Theorem 4.4.

COROLLARY 4.6. *Let X be a separable Banach space. The following are equivalent:*

(i) *X does not contain a subspace isomorphic to* ℓ_1.

(ii) *There exists an equivalent norm* $|||\cdot|||$ *on X such that if* (x_n) *is an* $f(n)$*-approximate* ℓ_1 *system with respect to* $|||\cdot|||$ *then* $f(n) > \delta n$ *for some* $\delta > 0$.

Our next goal is to use Theorem 4.4 to give examples of $f(n)$-approximate ℓ_1 systems which have no unconditional basic subsequences. To that end, we must recall the definition of the *mixed Tsirelson* spaces. Given a sequence $(\mathcal{M}_j)_{j=0}^{\infty}$ of compact families of finite subsets of $\mathbb{N}$, and given a sequence $(\theta_j)_{j=0}^{\infty}$ of real numbers converging to 0, the mixed Tsirelson space $T[(\mathcal{M}_j, \theta_j)_{j=0}^{\infty}]$ is defined in [2] as the completion of the linear space c_{00} under the norm $\|\cdot\|$ given as follows. For $x \in c_{00}$,

$$\|x\| = \max\left\{\|x\|_\infty, \sup_j \theta_j \sup\left\{\sum_{i=1}^{n} \|E_i x\| : (E_i)_{i=1}^{n} \text{ is } \mathcal{M}_j\text{-admissible}\right\}\right\},$$

where, for $E \subset \mathbb{N}$, Ex is the restriction of x to E and, for a family $\mathcal{M}$, an $\mathcal{M}$-admissible sequence $(E_i)_{i=1}^{n}$ is a sequence of subsets of $\mathbb{N}$ such that $E_1 < E_2 < \cdots < E_n$ and such that the set $\{\min E_1, \min E_2, \ldots, \min E_n\}$ belongs to $\mathcal{M}$.

In the definition of the spaces X and X_u below, $(\mathcal{M}_j)_{j=0}^{\infty}$ is an appropriate subsequence of $(\mathcal{S}_n)_{n=1}^{\infty}$, where $\mathcal{S}_n$ denotes the nth *Schreier family* (introduced in [1]) defined inductively as follows:

$$\mathcal{S}_0 = \{\emptyset\} \cup \{\{n\} : n \in \mathbb{N}\},$$

and, for $k \geqslant 0$,

$$\mathcal{S}_{k+1} = \{\emptyset\} \cup \left\{\bigcup_{i=1}^{n} A_i : n \in \mathbb{N},\, A_i \in \mathcal{S}_k,\, n \leqslant A_1 < A_2 < \cdots < A_n\right\}.$$

Let us also recall that an infinite-dimensional Banach space is *hereditarily indecomposable* (H.I.) if X does not have a subspace which can be expressed as a topological direct sum $Y \oplus Z$, with Y and Z infinite-dimensional. Observe that an H.I. space has no unconditional basic sequence. For suppose that $(b_n)_{n=1}^{\infty}$ is an unconditional basic sequence. Then the subspace generated by (b_n) can be decomposed as a direct sum of the subspaces generated by $(b_{2n})_{n=1}^{\infty}$ and by $(b_{2n-1})_{n=1}^{\infty}$

PROPOSITION 4.7. *The H.I. space X introduced in* [2] *has a spreading model isometrically equivalent to the unit vector basis of* ℓ_1.

Proof. We refer the reader to [2, p. 979] for the detailed definitions of the spaces X and X_u which are summarized below. Briefly, $X_u = T[(\mathcal{M}_j, 1/m_j)_{j=0}^{\infty}]$, where $\mathcal{M}_0 = \mathcal{S}_0$, $m_0 = 2$, and, for $j \geqslant 1$, $m_j > m_{j-1}^{m_{j-1}}$. In particular, $m_1 > 4$, which we use below.

These spaces are the completions of c_{00} equipped with norms defined by certain classes of linear functionals defined inductively as follows. For $j \geqslant 0$, set $K_j^0 = \{\pm e_n : n \in \mathbb{N}\}$. Assume that $\{K_j^n\}_{j=0}^{\infty}$ have been defined. Then we set $K^n = \bigcup_{j=0}^{\infty} K_j^n$, and for $j \geqslant 0$,

$$K_j^{n+1} = K_j^n \cup \left\{ \frac{1}{m_j}(f_1 + \cdots + f_d) : \bigcup_{i=1}^{d} \operatorname{supp} f_i (\operatorname{supp} f_1 < \cdots < \operatorname{supp} f_d) \right.$$

$$\left. \text{is } \mathcal{M}_j\text{-admissible and } f_1, \ldots, f_d \in K^n \right\}.$$

Set $A_j = \bigcup_{n=1}^{\infty} K_j^n$ $(j \geqslant 0)$ and $K = \bigcup_{n=0}^{\infty} K^n$. The norm $\|\cdot\|_u$ of X_u is defined thus:

$$\|x\|_u = \sup\{f(x) : f \in K\}.$$

To obtain X one defines certain sets $L_j^n \subset K_j^n$, $B_j = \bigcup_{n=1}^{\infty} L_j^n \subset A_j$, and $L = \bigcup_{j=0}^{\infty} B_j \subset K$. The norm $\|\cdot\|$ of X is defined thus:

$$\|x\| = \sup\{f(x) : f \in L\}.$$

An alternative definition of X (see [2, Remark 3.1]) is the following. For $x \in c_{00}$,

$$\|x\| = \max\left\{ \|x\|_\infty, \sup\left\{ \frac{1}{m_{2j}} \sum_{k=1}^{n} \|E_k x\| : j \geqslant 0, n \geqslant 1, \{E_1 < \cdots < E_n\} \right.\right.$$

$$\left.\left. \text{is } M_{2j}\text{-admissible} \right\}, \sup\left\{ |f(x)| : f \in \bigcup_{j=0}^{\infty} B_{2j+1} \right\} \right\}.$$

The construction in X is the same as the standard spreading model isometric to ℓ_1 in the Tsirelson space T. Indeed, let $e_i = x_{2i-1} + x_{2i}$, where (x_i) is the usual basis of X. Since $m_1 > 4$, the norm of e_i is achieved by partitions from the level $\mathcal{M}_0 = \mathcal{S}_1$, therefore $\|e_i\| = 1$. For any $k < n_1 < n_2 < \cdots < n_k$, we have that $\{2n_1 - 1, 2n_1, 2n_2 - 1, 2n_2, \ldots, 2n_k - 1, 2n_k\} \in \mathcal{S}_1$, whence $\|\sum_{i=1}^{k} x_{n_i}\| = k$. Therefore, the spreading model generated by (e_i) is isometric to ℓ_1. ∎

Remark 4.8. Odell and Schlumprecht [16] constructed spreading models isometric to ℓ_1 in T hereditarily, i.e., in every infinite dimensional subspace Y of T there exists (y_i) with spreading model 1-equivalent to the unit vector basis of ℓ_1. We can show a similar hereditary result for X_u and X. The proof requires more technical details from [2] and thus we chose not to include it here, since it is further away from the main topic of the present paper.

Remark 4.9. The H.I. space GM (introduced in [10]) has isometrically the same spreading models as the space S (introduced in [19]) [20]. S has a spreading model isomorphic to ℓ_1 [14]. Modifying the construction of [14] slightly shows that S has ℓ_1 isometrically as a spreading model [15].

COROLLARY 4.10. *Given $(f(n))$, there exists an H.I. Banach space X which has a basis (x_i) that is an $f(n)$-approximate ℓ_1 system. In particular, X does not contain any unconditional basic sequence.*

Remark 4.11. Konyagin and Temlyakov [13] defined a basis (x_i) to be *superdemocratic* if there exists a positive constant C such that, whenever $|A| = |B|$, then

$$\frac{1}{C}\left\|\sum_{i\in A}\eta_i x_i\right\| \leqslant \left\|\sum_{i\in B}\eta'_i x_i\right\| \leqslant C\left\|\sum_{i\in A}\eta_i x_i\right\|$$

for all choices of signs $(\eta_i)_{i\in A}$ and $(\eta'_i)_{i\in B}$. An example is given in [13] of a superdemocratic basis which is not unconditional. Note that the bases given by Corollary 4.10 are superdemocratic and their linear spans do not contain any unconditional basic sequence. In fact, provided $f(n) = o(n)$, we have

$$\lim_{n\to\infty}\ \sup_{|A|=|B|=n}\ \frac{\|\sum_{i\in A}\eta_i x_i\|}{\|\sum_{i\in B}\eta'_i x_i\|} = 1,$$

where the supremum is taken over all possible choices of signs.

5. $f(n)$-APPROXIMATE ℓ_1 SYSTEMS IN ℓ_1

In this section we construct some nontrivial examples of $f(n)$-approximate ℓ_1 systems in the space ℓ_1 itself. Actually, our construction can be carried out without much extra complication in any space that is isomorphic to ℓ_1. For any given $(f(n))$, we shall construct two examples of $f(n)$-approximate ℓ_1 systems: first, an unconditional basic sequence which is not equivalent to the ℓ_1-basis; secondly, a conditional basic sequence.

As motivation for these results let us recall that a sequence in an L_1 space is equivalent to the unit vector basis of ℓ_1 if it is, roughly speaking, "sufficiently disjoint." Here is one such criterion for sufficient disjointness which we state without proof.

PROPOSITION 5.1. *Let $(f_n)_{n=1}^\infty$ be a normalized sequence in an L_1 space. Suppose that there exist $t>0$ and a sequence $(A_n)_{n=1}^\infty$ of disjoint measurable sets such that*

$$\int_{A_n} |f_n| - \sum_{m\neq n} \int_{A_m} |f_n| \geqslant t \qquad (n\geqslant 1).$$

Then $\|\sum a_n f_n\| \geqslant t \sum |a_n|$ for all $(a_n)\in \ell_1$.

So our examples show that an $f(n)$-approximate ℓ_1 system, even for very slowly increasing $(f(n))$, does not necessarily possess enough disjointness to be equivalent to the ℓ_1 basis. It is ususally possible, on the other hand, to extract an ℓ_1 subsequence.

PROPOSITION 5.2. *Suppose that $k\in(0,1)$ and that $(x_i)_{i=1}^\infty$ is an $f(n)$-approximate ℓ_1 system in an L_1 space for $f(n)=kn$. Then $(x_i)_{i=1}^\infty$ has an ℓ_1 subsequence.*

Proof. Suppose, to derive a contradiction, that (x_i) has a weakly convergent subsequence. Then, passing to a subsequence, we may assume that $(y_i)_{i=1}^\infty$ is weakly null, where $y_i = x_{2i} - x_{2i-1}$. Since L_1 spaces have the *weak Banach–Saks property* [21], it follows that (after passing to a subsequence of (y_i) and relabelling) $(1/n)\sum_{i=1}^n y_i \to 0$. On the other hand,

$$\left\|\sum_{i=1}^n y_i\right\| = \left\|\sum_{i=1}^n (x_{2i}-x_{2i-1})\right\| \geqslant 2n - f(2n) = 2n(1-k),$$

which yields the contradiction. Finally, it is well-known that every sequence in an L_1 space which has no weakly convergent subsequence has an ℓ_1 subsequence. ∎

Let us recall the definition of the L_1-normalized Haar system on $[0,1]$. Let $h_0^0 \equiv 1$. For $n\geqslant 0$ and $1\leqslant k\leqslant 2^n$, we define h_k^n thus:

$$h_k^n = \begin{cases} 2^n & \text{on } [(k-1)/2^n, (2k-1)/2^{n+1}) \\ -2^n & \text{on } [(2k-1)/2^{n+1}, k/2^n) \\ 0 & \text{elsewhere.} \end{cases}$$

The dyadic Hardy space H_1 has the following norm:

$$\left\| a_0^0 h_0^0 + \sum_{n=0}^\infty \sum_{k=1}^{2^n} a_k^n h_k^n \right\| = \int_0^1 \left((a_0^0 h_0^0)^2 + \sum_{n=0}^\infty \sum_{k=1}^{2^n} (a_k^n h_k^n)^2 \right)^{1/2} dt.$$

Note that H_1 is isometrically isomorphic to a subspace of $L_1[0,1]$. To see this, observe that the mapping

$$f = a_0^0 h_0^0 + \sum_{n=0}^\infty \sum_{k=1}^{2^n} a_k^n h_k^n \mapsto (a_k^n h_k^n)_{n,k}$$

defines an isometric embedding from H_1 into $L_1(\ell_2(\mathbb{Z}\times\mathbb{Z}))$. Since $L_1[0,1]$ contains a subspace linearly isometric to ℓ_2, namely the closed linear span of a sequence of independent gaussian random variables, it follows that $L_1(\ell_2)$ is linearly isometric to a subspace of $L_1(L_1)$, which in turn is linearly isometric to $L_1[0,1]$.

Note also that each "layer" $(h_i^n)_{i=1}^{2^n}$ of the Haar system is isometrically equivalent to the unit vector basis of $\ell_1^{2^n}$:

$$\left\| \sum_{k=1}^{2^n} a_k h_k^n \right\| = \sum_{k=1}^{2^n} |a_k|. \tag{20}$$

PROPOSITION 5.3. *Given $(f(n))$, there exists an increasing sequence $(n_i)_{i=0}^{\infty}$ of nonnegative integers such that the "lacunary Haar system" $((h_j^{n_i})_{j=1}^{2^{n_i}})_{i=0}^{\infty}$ in H_1 is an $f(n)$-approximate ℓ_1 system.*

Proof. Set $n_0=0$. Suppose that $n_0<n_1<\cdots<n_k$ have been chosen so that $\mathscr{H}_k=((h_j^{n_i})_{j=1}^{2^{n_i}})_{i=0}^{k}$ satisfies

$$\left\| \sum_A \pm h_j^n \right\| > |A| - f(|A|) \tag{21}$$

for all $A\subseteq \mathscr{H}_k$ and for all choices of signs. Then there exists $\varepsilon_k>0$ such that

$$\left\| \sum_A \pm h_j^n \right\| > |A| - f(|A|) + \varepsilon_k \tag{22}$$

for all $A\subseteq \mathscr{H}_k$ and for all choices of signs. By a uniform integrability argument there exists $\delta>0$ such that if $\lambda(\bigcup_{(j,n)\in G} \operatorname{supp} h_j^n)<\delta$, where λ denotes Lebesgue measure, then for every $g=\sum_{(j,n)\in G} a_j^n h_j^n$, we have

$$\left\| g + \sum_A \pm h_j^n \right\| > \|g\| + \left\| \sum_A \pm h_j^n \right\| - \varepsilon_k$$

for all $A\subseteq \mathscr{H}_k$ and for all choices of signs. Select n_{k+1} so large that $f(\lfloor \delta 2^{n_{k+1}} \rfloor) > 2|\mathscr{H}_k|$, and let $\mathscr{H}_{k+1}=\mathscr{H}_k\cup(h_i^{n_{k+1}})_{i=1}^{2^{n_{k+1}}}$. Suppose that $A\subseteq\mathscr{H}_{k+1}$. We shall show that (21) is satisfied by A. Write $A=B\cup C$, where $B\subseteq\mathscr{H}_k$ and $C\subseteq(h_i^{n_{k+1}})_{i=1}^{2^{n_{k+1}}}$. There are two cases to consider. First, suppose that $|C|<\lfloor\delta 2^{n_{k+1}}\rfloor$. Then $\lambda(\operatorname{supp}(\sum_C \pm h_i^n)) = |C|\,2^{-n_{k+1}}<\delta$. Hence, by the choice of δ, we have

$$\begin{aligned}\left\| \sum_A \pm h_i^n \right\| &> \left\| \sum_B \pm h_i^n \right\| + \left\| \sum_C \pm h_i^n \right\| - \varepsilon_k \\ &> (|B| - f(|B|) + \varepsilon_k) + |C| - \varepsilon_k\end{aligned}$$

(by (22) and (20))

$$= |A| - f(|B|)$$
$$\geqslant |A| - f(|A|).$$

On the other hand, if $|C| \geqslant \lfloor \delta 2^{n_{k+1}} \rfloor$, then

$$\left\| \sum_A \pm h_i^n \right\| > \left\| \sum_C \pm h_i^n \right\| - \left\| \sum_B \pm h_i^n \right\|$$
$$\geqslant |C| - |B|$$
$$= |A| - 2\,|B|$$
$$\geqslant |A| - 2\,|\mathcal{H}_k|$$
$$> |A| - f(\lfloor \delta 2^{n_{k+1}} \rfloor)$$

(by the choice of n_{k+1})

$$\geqslant |A| - f(|A|). \quad \blacksquare$$

Remark 5.4. For $k = 1, 2, \ldots$, let $r_k = 2^{1-k} \sum_{i=1}^{2^{k-1}} h_i^{k-1}$ be the sequence of Rademacher functions. Then (r_k) is equivalent in H_1 to the unit vector basis of ℓ_2 and its span is complemented by the orthogonal projection. Since the linear span of every lacunary Haar system contains a subsequence of the Rademacher functions, it follows that the closed linear span of a lacunary Haar system contains a complemented block subspace isomorphic to ℓ_2.

We shall now transfer the above example from H_1 to ℓ_1 by a localization argument.

THEOREM 5.5. *Suppose that X is isomorphic to ℓ_1. Then, given $(f(n))$ and $\alpha > 1$, X contains a normalized basic sequence (y_i) satisfying the following:*

(i) *(y_i) is an $(f(n))$-appproximate ℓ_1 system.*

(ii) *(y_i) is α-unconditional.*

(iii) *For $k = 1, 2, \ldots$, the unit vector basis of ℓ_2^k is uniformly equivalent to a uniformly complemented block basis of (y_i) (i.e., the norms of the projections are uniformly bounded). (This implies, in particular, that $[y_i]$ is not isomorphic to ℓ_1 since ℓ_1 does not contain uniformly complemented uniformly equivalent copies of ℓ_2^k.)*

The proof of Theorem 5.5 requires two technical lemmas. First let us recall that if X and Y are two n-dimensional normed spaces, then their *Banach–Mazur distance* $d(X, Y)$ is defined thus:

$$d(X, Y) = \inf\{\|T\|\,\|T^{-1}\| : T : X \to Y \text{ is an isomorphism}\}.$$

LEMMA 5.6. *Let* $(f(j))_{j=1}^{\infty}$, $\alpha > 1$, *and* $k \in \mathbb{N}$ *be given. There exist* $m \in \mathbb{N}$, $n \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that, whenever* $d(X, \ell_1^n) < 1+\varepsilon$, *then there exist* m *unit vectors* $(x_i)_{i=1}^m$ *in* X *satisfying the following:*

(i) $(x_i)_{i=1}^m$ *is an* $f(j)$*-approximate* ℓ_1 *system.*

(ii) $(x_i)_{i=1}^m$ *is an* α*-unconditional basic sequence.*

(iii) ℓ_2^k *is uniformly equivalent to a uniformly complemented block basis of* $(x_i)_{i=1}^m$.

Proof. By Proposition 5.3, there is a lacunary Haar system which is an $f(n)$-approximate ℓ_1 system. Let $(y_i)_{i=1}^m$ be an enumeration of the the first k layers of this system. Then $(y_i)_{i=1}^m$ is 1-unconditional and satisfies conditions (i) and (iii). Since H_1 is isometric to a subspace of L_1, given $\eta > 0$, we can find a positive integer n such that $[y_i]_{i=1}^m$ is $(1+\eta)$-isomorphic to a subspace of ℓ_1^n. The lemma now follows by a standard perturbation argument. ∎

The following lemma is implicit in James's proof that ℓ_1 is not distortable [12].

LEMMA 5.7. *Let* $(\varepsilon_i)_{i=1}^{\infty}$ *and* $(\alpha_i)_{i=1}^{\infty}$ *be sequences of positive numbers, and let* $(n_i)_{i=1}^{\infty}$ *be a sequence of positive integers. Let* X *be isomorphic to* ℓ_1. *There exist subspaces* $F_i \subseteq X$ *satisfying the following:*

(i) $d(F_i, \ell_1^{n_i}) < 1+\varepsilon_i$;

(ii)

$$\left\| \sum_{i=n}^{\infty} x_i \right\| \geqslant (1-\alpha_n) \sum_{i=n}^{\infty} \|x_i\|$$

for all $n \geqslant 1$ *and all* $x_i \in F_i$.

Proof of Theorem 5.5. It will be clear from the construction (and from Lemmas 5.6 and 5.7) that the sequence (y_i) can always be chosen to be α-unconditional provided $\alpha > 1$. Therefore, to avoid unnecessary complication, we shall omit the verifiication of (ii).

Select $\beta > 0$ and a positive sequence $((g(n))$ such that

$$\beta + 3g(n) = f(n) \qquad (n \geqslant 1).$$

Fix $k \geqslant 1$ and let n_k, m_k, and ε_k be as given by Lemma 5.6 when applied to the sequence $(g(n)/2^k)_{n=1}^{\infty}$. Now choose positive integers $p_m \uparrow \infty$ $(m \geqslant 1)$ such that

$$g(n_1 + n_2 + \cdots + n_{p_m}) > 2(n_1 + n_2 + \cdots + n_m). \tag{23}$$

Next choose positive numbers α_i $(i \geqslant 1)$ such that

$$(n_1 + n_2 + \cdots + n_{p_{i+1}})\, \alpha_i < \beta. \tag{24}$$

Apply Lemma 5.7 to (ε_i), (α_i), and (n_i), to find subspaces F_i satisfying the conclusion of the lemma.

By Lemma 5.6, for each $k \geqslant 1$ there exist vectors $(x_i^k)_{i=1}^{m_k}$ in F_k satisfying (i)–(iii) of Lemma 5.6 (applied to $(g(n)/2^k)_{n=1}^{\infty}$). Let $(y_i)_{i=1}^{\infty}$ be the enumeration of the sequence $x_1^1, \ldots, x_{m_1}^1, x_1^2, \ldots, x_{m_2}^2, \ldots$, and for $k \geqslant 1$, let $B_k = \{i \colon y_i \in F_k\}$.

By Lemma 5.6, it is clear that (y_i) satisfies condition (iii) of Theorem 5.5. So it remains only to verify that (y_i) is an $f(n)$-approximate ℓ_1 system. Suppose that $A \subset \mathbb{N}$ and that $N = |A|$ satisfies

$$n_1 + n_2 + \cdots + n_{p_m} < N \leqslant n_1 + n_2 + \cdots + n_{p_{m+1}}.$$

(The verification is similar but easier if $N \leqslant n_1 + \cdots + n_{p_1}$.) Then

$$\left\| \sum_{i \in A} \pm y_i \right\| \geqslant \left\| \sum_{j=m+1}^{\infty} \left(\sum_{i \in A \cap B_j} \pm y_i \right) \right\| - (n_1 + \cdots + n_m)$$

(by the triangle inequality since $|B_j| = m_j \leqslant n_j$)

$$\geqslant (1 - \alpha_m) \sum_{j=m+1}^{\infty} \left\| \sum_{i \in A \cap B_j} \pm y_i \right\| - (n_1 + \cdots + n_m)$$

(by Lemma 5.7)

$$\geqslant (1 - \alpha_m) \sum_{j=m+1}^{\infty} \left(|A \cap B_j| - \frac{g(|A \cap B_j|)}{2^j} \right) - (n_1 + \cdots + n_m)$$

(since $(y_i)_{i \in B_j}$ is a $(g(n)/2^j)$-approximate ℓ_1 system)

$$\geqslant (1 - \alpha_m) \left(\sum_{j=1}^{\infty} |A \cap B_j| - (n_1 + \cdots + n_m) \right)$$

$$- g(|A|) \left(\sum_{j=1}^{\infty} 2^{-j} \right) - (n_1 + \cdots + n_m)$$

$$\geqslant (1 - \alpha_m)\, |A| - g(|A|) - 2(n_1 + \cdots + n_m)$$

$$\geqslant |A| - \beta - g(|A|) - 2(n_1 + \cdots + n_m)$$

(by (24) since $|A| = N \leqslant n_1 + n_2 + \cdots + n_{p_{m+1}}$.)

$$\geqslant |A| - \beta - 3g(|A|)$$

(by (23) since $|A| = N > n_1 + \cdots + n_{p_m}$)

$$> |A| - f(|A|)$$

by the choice of β and $(g(n))$. This proves that (y_n) is an $f(n)$-approximate ℓ_1 system. ∎

We obtain a strengthening of Theorem 4.4.

COROLLARY 5.8. *Let X be a Banach space. The following are equivalent:*

(i) *X has a spreading model that is isometrically eqivalent to the unit vector basis of ℓ_1.*

(ii) *For all $(f(n))$, X contains an $f(n)$-approximate ℓ_1 basic sequence whose closed linear span is not isomorphic to ℓ_1.*

Proof. When X does not contain ℓ_1, the result is an immediate consequence of Theorem 4.4. When X does contain ℓ_1, the result follows from (iii) of Theorem 5.5. ∎

We can also consider the Haar system in L_1 instead of H_1. Let us observe that every lacunary Haar system in L_1 is a *conditional* monotone basis for its linear span. The conditionality follows from the easily verified fact that the full Haar system is equivalent to a block basis of every lacunary Haar system. All the proofs of this section go through mutatis mutandis for the lacunary Haar system in L_1 to yield the following analogue of Theorem 5.5.

THEOREM 5.9. *Suppose that X is isomorphic to ℓ_1. Then, given $(f(n))$ and $\alpha > 1$, X contains a normalized basic sequence (y_n) satisfying the following:*

(i) *(y_n) is an $(f(n))$-appproximate ℓ_1 system.*

(ii) *(y_n) is a conditional basis for its closed linear span with basis constant at most α.*

(iii) *The unit vector basis of ℓ_2^k $(k \geqslant 1)$ is uniformly equivalent to a block basis of (y_n).*

If $X = \ell_1$, we may also take (y_i) to be a monotone basic sequence.

6. LACUNARY HAAR SYSTEMS ARE QUASI-GREEDY

Let $(x_n)_{n=1}^{\infty}$ be a normalized basis for X with biorthogonal functionals $(x_n^*)_{n=1}^{\infty}$. For each $x \in X$ and $m = 1, 2, \ldots$ we define

$$\mathcal{G}_m(x) = \sum_{n \in A} x_n^*(x)\, x_n,$$

where A is a set of cardinality m such that $\min\{|x_n^*(x)| : n \in A\} \geqslant \max\{|x_n^*(x)| : n \in \mathbb{N} \setminus A\}$. Note that A is not necessarily uniquely defined. We say that the basis (x_n) is *quasi-greedy* if $\mathcal{G}_m(x) \to x$ for each $x \in X$.

THEOREM 6.1. *Let $\varepsilon > 0$. There exists an increasing sequence of integers $(n_j)_{j=0}^{\infty}$ such that the lacunary Haar system $((h_j^{n_i})_{j=1}^{2^{n_i}})_{i=0}^{\infty}$ in $L_1[0,1]$ is a quasi-greedy basis satisfying $\|\mathcal{G}_m(x)\| \leqslant (1+\varepsilon)\,\|x\|$ for all x in its closed linear span and for all $m \geqslant 1$.*

The proof of Theorem 6.1 uses two auxiliary results. The first is an obvious symmetry property of the Haar system.

LEMMA 6.2. *Every reaarangement of the Haar system which merely changes the order of terms within each layer of the Haar system (i.e., so that every h_i^k on layer k comes before every h_i^{k+1} on layer $k+1$ after the rearrangement) is a monotone basis.*

Proof. Clearly, every such rearrangement of the Haar system is a martingale difference sequence with respect to the standard dyadic filtration. ∎

For a proof of the following we refer the reader to [22].

THEOREM A. *Let (x_n) be a basis for the Banach space X. The following are equivalent:*

(i) *(x_n) is a quasi-greedy basis.*

(ii) *There exists a constant C such that $\|\mathcal{G}_m(x)\| \leqslant C\,\|x\|$ for all x in the linear span of (x_n) and for all $m \geqslant 1$.*

Proof of Theorem 6.1. Select $\varepsilon_i \downarrow 0$ such that $\prod_{i=1}^{\infty} (1+\varepsilon_i) < 1+\varepsilon$. Set $n_0 = 1$ and suppose that $n_0 < n_1 < \cdots < n_k$ have been chosen. Let $(x_i)_{i=1}^{N_k}$ be the lexicographical ordering of the elements of $\bigcup_{j=0}^{k} \bigcup_{i=1}^{2^{n_j}} h_i^{n_j}$ and let $F_k = [x_i]_{i=1}^{N_k}$. We shall assume as an inductive hypothesis that

$$\|\mathcal{G}_m(x)\| \leqslant \left(\prod_{i=1}^{k} (1+\varepsilon_i) \right) \|x\| \qquad (x \in F_k). \tag{25}$$

Let $\delta_{k+1} = \varepsilon_{k+1}/N_k$. By uniform integrability there exists $\alpha_{k+1} > 0$ such that if $x \in F_k$ then

$$\|x+y\| \geqslant \frac{1}{1+\varepsilon_{k+1}}(\|x\|+\|y\|) \tag{26}$$

whenever $y \in L_1$ satisfies $\lambda(\operatorname{supp} y) < \alpha_{k+1}$.

Choose n_{k+1} such that $2^{1-n_{k+1}}/\delta_{k+1} < \alpha_{k+1}$. Fix $m \geqslant 1$. Suppose that $\|x+y\| = 1$, where $x \in F_k$ and $y \in [h_i^{n_{k+1}}]_{i=1}^{2^{n_{k+1}}}$. Note that $\|y\| \leqslant 2$ (since the Haar system is monotone) and that

$$\mathscr{G}_m(x+y) = \mathscr{G}_{m_1}(x) + \mathscr{G}_{m_2}(y),$$

for some m_1, m_2 with $m = m_1 + m_2$. We now consider two cases. First suppose that the smallest nonzero coefficient in the basis expansion of $\mathscr{G}_m(x+y)$ has absolute value at least δ_{k+1}. Since $\lambda(\operatorname{supp}(h_i^{n_{k+1}})) = 2^{-n_{k+1}}$, we have

$$\lambda(\operatorname{supp} \mathscr{G}_{m_2}(y)) < \frac{\|y\|}{2^{n_{k+1}}\delta_{k+1}} \leqslant \frac{2}{2^{n_{k+1}}\delta_{k+1}} < \alpha_{k+1}.$$

Thus

$$\begin{aligned}\|\mathscr{G}_m(x+y)\| &\leqslant \|\mathscr{G}_{m_1}(x)\| + \|\mathscr{G}_{m_2}(y)\| \\ &\leqslant \left(\prod_{i=1}^{k}(1+\varepsilon_i)\right)(\|x\| + \|\mathscr{G}_{m_2}(y)\|)\end{aligned}$$

(by (25))

$$\leqslant \left(\prod_{i=1}^{k+1}(1+\varepsilon_i)\right)\|x + \mathscr{G}_{m_2}(y)\|$$

(by (26))

$$\leqslant \left(\prod_{i=1}^{k+1}(1+\varepsilon_i)\right)\|x+y\|,$$

where the last line follows from Lemma 6.2. For the second case we assume that the smallest nonzero coefficient in the basis expansion of $\mathscr{G}_m(x+y)$ has absolute value at most δ_{k+1}. Then $|x_i^*(x - \mathscr{G}_{m_1}(x))| \leqslant \delta_{k+1}$ for $1 \leqslant i \leqslant N_k$. Thus, by the choice of δ_{k+1}, we have

$$\|x - \mathscr{G}_{m_1}(x)\| \leqslant N_k \delta_{k+1} < \varepsilon_{k+1}.$$

Thus

$$\begin{aligned}\|\mathcal{G}_m(x+y)\| &= \|\mathcal{G}_{m_1}(x)+\mathcal{G}_{m_2}(y)\| \\ &\leqslant \|x+\mathcal{G}_{m_2}(y)\|+\|x-\mathcal{G}_{m_1}(x)\| \\ &\leqslant \|x+y\|+\varepsilon_{k+1}\end{aligned}$$

(by Lemma 6.2)

$$\begin{aligned}&= (1+\varepsilon_{k+1})\,\|x+y\| \\ &\leqslant \left(\prod_{i=1}^{k+1}(1+\varepsilon_i)\right)\|x+y\|,\end{aligned}$$

which establishes the inductive hypothesis for $k+1$. Thus (x_n) satisfies (ii) of Theorem A, with $C=\prod_{i=1}^{\infty}(1+\varepsilon_i)<1+\varepsilon$. ∎

Remark 6.3. Recall that the full Haar system is *not* a quasi-greedy basis of L_1. To see this most easily, fix $n\geqslant 1$ and $\varepsilon>0$. Let $x_n=h_0^0+\sum_{k=0}^{n-1}((1+\varepsilon)\,h_1^{2k}+h_1^{2k+1}))$. Then, for sufficiently small ε, we have $\|x_n\|\leqslant 2$. But $\mathcal{G}_n(x)=\sum_{k=0}^{n-1}((1+\varepsilon)\,h_1^{2k})$, so $\|\mathcal{G}_n(x)\|\geqslant n/4$. Since n is arbitrary, Theorem A implies that the Haar system is not quasi-greedy.

Remark 6.4. The "dual" version of Theorem 6.1 is false. The L_∞-normalized Haar system is a basis for its closed linear span in L_∞. However, $(h_1^{n_k})$ is equivalent to the summing basis of c_0 for every subsequence (n_k). It is easy to see that the summing basis is not quasi-greedy.

For our final result, let us recall the definition of the *best m-term approximation*. For $x\in X$ and $m=0, 1, \ldots$ we set

$$\sigma_m(x)=\inf\{\|x-\Sigma_{n\in A}a_n x_n\|: |A|\leqslant m\}.$$

Then the error of the greedy algorithm as compared to the error in the best m-term approximation is measured by the following quantity [22]:

$$e_m=\sup_{x\in X}\frac{\|x-\mathcal{G}_m(x)\|}{\sigma_m(x)}\qquad\left(\text{with }\frac{0}{0}=1\right).$$

THEOREM 6.5. *Given an unbounded increasing sequence $(f(n))$, with $f(1)\geqslant 73$, there exists a lacunary Haar system (x_n) in L_1 such that $e_n\leqslant f(n)$ for all $n\in\mathbb{N}$.*

The proof of Theorem 6.5 uses the following result which is a special case of [22, Theorem 5]. We refer the reader to [22] for the proof.

THEOREM B. *Let $(g(n))$ be a positive increasing sequence. Suppose that (x_n) is a normalized basis for X such that*

$$\frac{1}{g(|A|)} \sum_{n \in A} |a_n| \leqslant \left\| \sum_{n \in A} a_n x_n \right\| \leqslant \sum_{n \in A} |a_n|,$$

for all finite $A \subset \mathbb{N}$ and scalars (a_n). Then $e_n \leqslant 3g(n)+1$ for all n.

Proof of Theorem 6.5. Let $g(n) = (f(n)-1)/72$. Then $(g(n))$ is an unbounded increasing sequence with $g(1) \geqslant 1$. By the analogue of Proposition 5.3 for the L_1 norm there exists a lacunary Haar system $(x_n)_{n=1}^{\infty}$ which is a $g(n)$-approximate ℓ_1 system. By Proposition 2.5 ,

$$\frac{1}{24g(|A|)} \sum_{n \in A} |a_n| \leqslant \left\| \sum_{n \in A} a_n x_n \right\| \leqslant \sum_{n \in A} |a_n|.$$

Thus, by Theorem B, $e_n \leqslant 72g(n)+1 = f(n)$. ∎

Remark 6.6. Theorem 6.5 is essentially best possible since (e_n) is a bounded sequence only if (x_n) is an unconditional basis [13].

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